PHYS4031 STATISTICAL MECHANICS

Problem Set 7 turned (EXTRA) SAMPLE QUESTIONS - 1 December 2016

In view of many of you have to prepare for the final-year project presentation next week, there will not be a Problem Set 7. Originally, I intended to give you a problem on the low-temperature physics of ultra-relativistic Fermi gas and how the results can be applied in astrophysical contexts. If you are taking the astrophysics course next term, you may see the same problem again.

TA will post solutions to SQ32,33 no later than 5 December 2016.

PHYS4031 ANNOUNCEMENT ON FINAL EXAM

Coverage: Chapter I to Chapter XIII (end of Ideal Fermi Gas chapter), including all materials discussed in class notes, lectures, sample questions in exercise classes, and problem sets. Sections and appendices in class notes marked "Optional" are excluded.

Time/Venue: Arranged centrally by University. Check time/venue yourself.

SQ32 - Low-temperature physics of 3D ultra-relativistic fermions

SQ33 - $T = 0$ ultra-relativistic fermions degenerate pressure and the fate of a star

SQ32 *Low-temperature physics of 3D ultrarelativistic Fermi Gas.*

In SQ30, TA worked out the $T = 0$ physics of a 3D ultra-relativistic Fermi gas. Here, the low-temperature physics is explored.

(a) Using the Sommerfeld expansion, work out how the chemical potential μ shifts at low temperatures.

(b) Hence, work out the energy $E(T)$ at low temperatures and find the heat capacity C_V and identify its temperature dependence.

(c) Give a qualitative hand-waving argument for the results in part (b).

SQ33 (*A bit of astrophysics and Fermi Gas: Degenerate pressure and stars ran out of fuel.*) (See Ch.XIII, SQ30, SQ32)

Many of you like astrophysics. Here is a short chapter on astrophysics. Stars burn by fusion. However, after a long time, it will run out of fuel, e.g., hydrogen. The star then becomes a system full of electrons (and other heavier particles, etc.). With no more fusion, gravity tends to make the star collapse. However, quantum mechanics (ideal Fermi gas) could save the star from collapsing! Let's see how (and if) it works. For simplicity (and in fact it is quite OK, why?), we will make use of $T = 0$ physics.

(a) In Ch.XIII, we showed that the pressure of a 3D ideal non-relativistic Fermi gas at $T = 0$ is given by:

$$p \propto n^{5/3} \propto \left(\frac{N}{V}\right)^{5/3} .$$

Referring to the full expression of p (see class notes), point out that p is inversely proportional to the electron mass. Hence, argue that electrons are the most effective in providing a pressure opposing the gravitational pull. (This is why we don't consider the contributions from other particles that may also be present, when many electrons are around.)

(b) Let ρ be the mass density of a star. Let's say the atoms in the star have Z electrons (atomic number) per atom and a mass number A and thus a mass Am_p per atom, where m_p is the mass of a proton (also close to that of neutron). Show that

$$\frac{N}{V} \approx \frac{Z\rho}{Am_p} .$$

and hence the pressure due to the gas of electrons is

$$p_{electron} \propto \rho^{5/3}.$$

This pressure $p_{electron}$ due to the Pauli exclusion principle is an outward pressure that tends to oppose the gravitational pull.

(c) The above relation is important, as it determines the fate of a dying star. As the star shrinks in size due to gravitational pull, the mass does not change but the mass density ρ increases and thus $p_{electron}$ increases as $\propto \rho^{5/3}$. **Hopefully**, the increase in $p_{electron}$ will be sufficient to oppose the pressure due to gravitational pull when the star shrinks to a certain radius. Then there will be a stable situation, called a **white** (meaning "hot", but we know that low-temperature physics is sufficient, since we studied stat mech) **dwarf** (meaning small), i.e., white dwarf. For the gravitation potential energy, it is easy to guess that it must go like $\sim M^2/R \sim M^2/V^{1/3}$, what else can it be! More accurately, like the problem of a uniformly charged sphere (see Griffiths' textbook on electrodynamics), the gravitational potential energy of a uniform sphere of radius R and mass M is given by

$$\mathcal{U}_{gravity} = -\frac{3}{5}\frac{GM^2}{R}.$$

It can be shown (by mechanics and thermodynamics) that gravity exerts an inward pressure of magnitude proportional to $\mathcal{U}_{gravity}/V$. More precisely

$$p_{gravity} = \frac{\mathcal{U}_{gravity}}{3V}.$$

We only need the magnitude of this pressure and know that it is an inward pressure. Show that

$$p_{gravity} \propto \rho^{4/3}.$$

(d) Now, argue that it is possible to achieve a stable situation as a star shrinks. Note that this stability relies heavily on the different powers in ρ for $p_{electron}$ and $p_{gravity}$. Derive an expression for ρ_0 in terms of M, A, and Z, where ρ_0 is the mass density at which the two pressures balance each other. Hence, show that the radius of the resulting white dwarf R_0 is related to the mass of the star by $R_0 \propto M^{-1/3}$. [Remark: Thus, **if** the electrons can be treated as non-relativistic particles, then the fate of a star is to become a white dwarf. But this is a big "**if**"!]

(e) **What could go wrong?** We discussed that for the electron number density N/V in a metal, E_F is about a few eV. For electrons, $mc^2 = 0.511\ MeV$, which is $\approx 10^6\ eV$. Here is the point! We still have $E_F = \hbar^2 k_F^2/2m \ll mc^2$ in metals and thus we can treat the electrons in metals as non-relativistic particles. We also made this assumption in the discussion above. One could imagine that in a star (very massive and not so big), the number density N/V will be very large. In k-space, the allowed states are still the dots in k-space (that come from fitting waves to a box). A higher N/V implies that we will fill single-particle states up to a much higher Fermi wavevector k_F. [Important: This statement about filling states in k-space is TRUE for both non-relativistic and relativistic particles.] Sooner or later, we will have $\hbar^2 k_F^2/2m \gg mc^2$ and we can no longer assume non-relativistic electrons. Like any good physicist, we always try the simplest possible path – treating the electrons as ultrarelativistic.
In SQ30, it was shown that ultrarelativistic electrons, i.e., $\epsilon = c\hbar k$, give a $T = 0$ pressure $p \propto n^{4/3}$. Hence, we also have the pressure $p_{ultra-rel-electron}$ due to ultrarelativistic electrons behaves like

$$p_{ultra-rel-electron} \propto \rho^{4/3}.$$

Comparing the result with $p_{gravity}$ comes a bad news! For ultrarelativistic electrons, the pressure increases with ρ in a similar fashion with $p_{gravity}$ (see that - same "4/3" power). Thus, as the star shrinks, while $p_{ultra-rel-electron}$ increases, $p_{gravity}$ also increases and it is always bigger than the electron pressure. Thus, electrons alone will not be sufficient to counteract gravity for stars with N/V higher than some value. Astrophysicists don't use N/V, they say that for stars with masses higher than a certain value, then they cannot become white dwarfs. Let's estimate that mass. Let's say when $\hbar k_F > mc$, we must treat the electrons as ultrarelativistic. Turn this into an inequality for N/V and hence for ρ.

(f) Here is an estimation. Let's use the expression for ρ_0 in part (d) and substitute it into the inequality in part (e). Obtain an inequality for M, i.e., $M >$ *something*, so that a star with a large mass, a white dwarf is not stable. For $Z/A \sim 1/2$ or 1, express that (*something*) in terms of the mass of the Sun ($M_\odot = 1.99 \times 10^{30}\ kg$). In astrophysics (with more proper treatment), the number is $1.44\ M_\odot$. That is to say, we don't expect to observe white dwarfs with mass larger than $1.44 M_\odot$. It also says that our Sun will eventually become a white dwarf. Cool! This limit for the mass of white dwarfs is called the Chandrasekhar limit. Chandrasekhar (1910-1995) was awarded the 1983 Nobel Physics Prize for his work in white dwarfs.

(g) Don't need to do anything. Read the whole problem again. It is about degenerate pressure of electrons and its relevance to astrophysics. Neutrons are also fermions, they can also exert a pressure to opposite gravity. It is the case in neutron stars. You might have read many popular science books on astrophysics in the past. Now, as a graduating physics student, see if you can construct a story (or a talk) on the fate of a star of considerable rigor in physics and yet understandable to high school students. A harder generalization of the problem is to work on a gas of relativistic (instead of ultra-relativistic) Fermi Gas.

Finally, this ends the course. Our coverage is in par with many courses in good research universities in the world. If you follow what we discussed, you could claim that you have learnt the materials in the first 8 chapters of Pathria's book *Statistical Mechanics*, which is a popular graduate level textbook; and the first 11 chapters of McQuarrie's book *Statistical Mechanics*, which is also a popular graduate text with a flavor of physical chemistry.

See SQ31 for a Summary of the Course. See also Chapter XV and the references therein.

[Remark: Access to the Course page will be closed on 28 December 2016.]